STUDENT'S NAME: ____________________

TEACHER'S NAME: ____________________

BAULKHAM HILLS HIGH SCHOOL

.

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2007

MATHEMATICS

EXTENSION 2

GENERAL INSTRUCTIONS:

- Reading time − 5 minutes.
- Working time − 3 hours.
- Write using blue or black pen.
- Start each question on a new page.
- Write your Student Number at the top of each page.
- Calculators may be used.
- A table of standard integrals is provided.
- **ALL** necessary working should be shown in every Question.

QUESTION 1 (15 marks)

Marks

(a) Simplify $\sin(A-B)+\sin(A+B)$ and hence find

$$\int \sin 5x \cos 3x \, dx$$ 3

(b) Find real constants A, B, C such that:

$$\frac{x^2+5x+2}{(x^2+1)(x+1)} \equiv \frac{\mathrm{Ax}+\mathrm{B}}{x^2+1}+\frac{C}{x+1}$$

and hence find $\displaystyle\int \frac{x^2+5x+2}{\left(x^2+1\right)\left(x+1\right)}dx$ 3

(c) Find $\int \sin^{-1} x \, dx$ 3

(d) Find $\displaystyle\int \sqrt{\frac{1+x}{1-x}}dx$ 3

(e) Evaluate $\displaystyle\int_0^1 \mathrm{x}^5 e^{x^3} \mathrm{dx}$ 3

QUESTION 2 (15 marks)

(a) If $z = 2\sqrt{3}\,i - 2$ find:

(i) $|z|$ 1

(ii) $\arg z$ 1

(iii) $\mathrm{Re}\,(1+2\mathrm{i})\,\overline{\mathrm{z}}$ 2

b) If ω is a complex root of the equation

$$\mathrm{z}^3 = 1:$$

(i) Show that $1 + \omega + \omega^2 = 0$ 2

(ii) Find the value of $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$ 3

(c) Find the equation of the locus of z where $|z - 2i| = \mathrm{Im}\, z$. 3

(d) Given $z = 2 - i$, find real values of a and b such that: 3

$$\mathrm{a\,z} + \frac{\mathrm{b}}{\mathrm{z}} = 1$$

QUESTION 3 (15 marks)

Marks

(a) Sketch the curve:

$f(x) = \dfrac{(x+1)(x+3)}{x}$ showing the stationary points and asymptotes. 3

Hence draw neat half page sketches of:

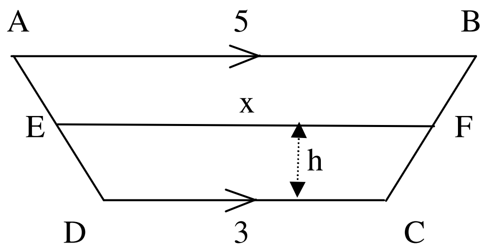
(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y = |f(x)|$ 1

(iii) $y = \sqrt{f(x)}$ 1

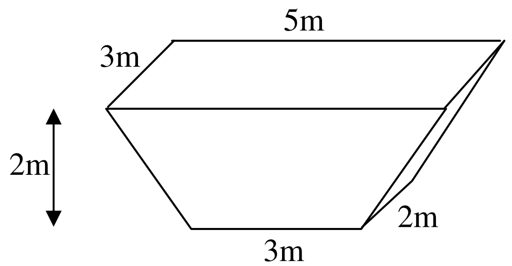
(iv) $y = \log_e f(x)$ 2

(b)



A trapezium ABCD has parallel sides AB = 5 m and CD = 3m, 2m apart. E lies on AD and F lies on BC such that EF is parallel to DC. The distance from EF to DC is h m and EF = x show that:

$x = 3 + h$. 1



NOT TO SCALE

The diagram is of a waste bin with a rectangular base of side 3m and 2m. Its top is also rectangular, parallel to the base with dimensions 5m and 3m. The bin has a depth of 2m, each of its four sides are trapeziums. Find the volume of the bin. 5

QUESTION 4 (15 marks)

Marks

(a) (i) Prove that the normal at point $P\left(cp, \frac{c}{p}\right)$ on the curve $xy = c^2$

is $p^3x - py = c\left(p^4 - 1\right)$ 2

(ii) The normal at P meets the hyperbola again at point $Q\left(\text{cq}, \frac{c}{q}\right)$. Prove that 2

$p^3q = -1$

(iii) The tangent at P meets the y axis at R. 3

Show that the area of the triangle PQR is:

$$A = \frac{c^2}{2}\left(p^2 + \frac{1}{p^2}\right)^2$$

and hence find the minimum area of this triangle. 2

(b) (i) Evaluate $\int_0^{\frac{\pi}{2}} (\sin t)^{2k} \cos t \, dt$ 1

(ii) Noting that $(\cos t)^{2n+1} = \cos t\left(1 - \sin^2 t\right)^n$ and by using the Binomial Theorem to expand $\left(1 - \sin^2 t\right)^n$ where n is a positive integer, show that

$$\int_0^{\frac{\pi}{2}} (\cos t)^{2n+1} dt = \sum_{r=0}^{n} (-1)^r \frac{1}{2r+1} \cdot {}^nC_r$$ 3

(iii) Use the result of part (ii) to evaluate

$$\int_0^{\frac{\pi}{2}} \cos^7 t \, dt$$ 2

QUESTION 5 (15 marks)

Marks

(a) The quadratic equation $x^2 - x + k = 0$ where k is a real number has 2 distinct positive roots α *and* β.

Show that:

(i) $0 < k < \frac{1}{4}$ 2

(ii) $\alpha^2 + \beta^2 = 1 - 2k$ and hence deduce that $\alpha^2 + \beta^2 > \frac{1}{2}$ 3

(iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$ 2

(b) Let α, β and γ be the roots of:

$x^3 - 7x^2 + 18x - 7 = 0$

Find the polynomial with roots:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ 2

(ii) $(1+\alpha^2), (1+\beta^2), (1+\gamma^2)$ 3

(c) By considering the stationary values of:

$f(x) = x^3 - 3px^2 + 4q$, where p and q are positive real constants, show

that the equation $f(x) = 0$ has 3 real distinct roots if

$p^3 > q.$ 3

QUESTION 6 (15 marks)

Marks

(a) A particle of mass m is moving vertically in a resisting medium in which the resistance to the motion has a magnitude of $\frac{1}{10}$ m v^2 where the particle has speed u ms^{-1}. The acceleration due to gravity is g ms^{-2}.

(i) If the particle falls vertically downwards from rest, show that its acceleration is given by:

$$a = g - \frac{1}{10}v^2.$$

Hence show that its terminal speed V ms^{-1} is given by

$$V = \sqrt{10g}\ .$$ 2

(ii) If the particle is projected vertically upwards with speed $V\tan\alpha\, ms^{-1}$ $(0 < \alpha < \frac{\pi}{2})$ show that its acceleration $a\,ms^{-2}$ is given by

$$a = -\left(g + \frac{v^2}{10}\right)$$

Hence show that it reaches a maximum height H metres given by :

$$H = 5\log_e \sec^2\alpha$$ 4

and that it returns to its point of projection with speed

$$V\sin\alpha\, ms^{-1}.$$ 4

(b) The roots of the equation $4x^3 - 36x^2 + 107x + k = 0$ are in arithmetic progression, find:

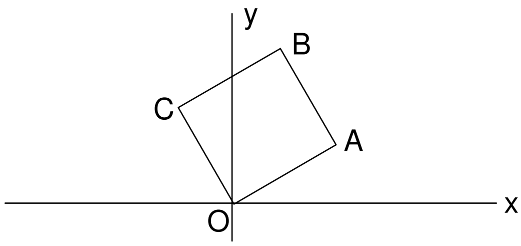
(i) k 2

(ii) the roots of the equation. 3

QUESTION 7 (15 marks)

Marks

(a) In the square OABC shown below, the point A represents $4 + 3i$. What complex numbers do the points B and C represent.



3

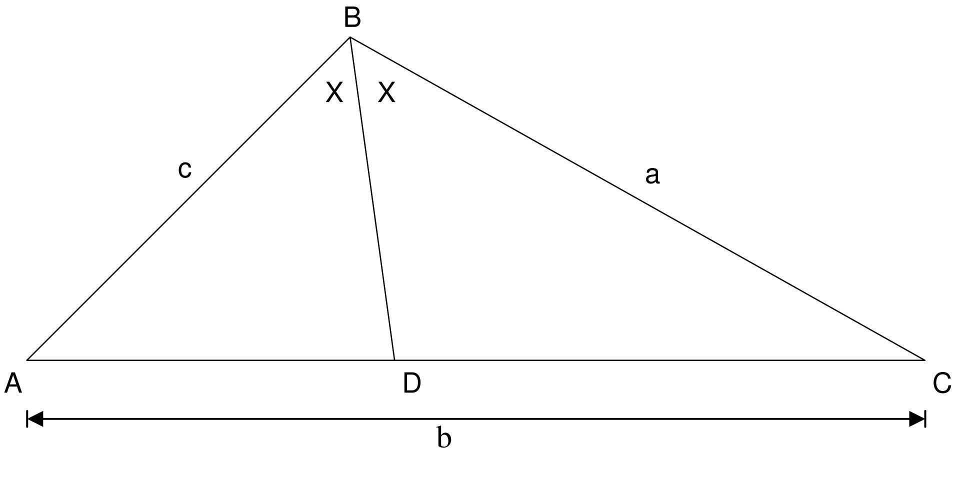
(b) (i) Determine the real values of k for which the equation:

$$\frac{x^2}{19 - k} + \frac{y^2}{7 - k} = 1$$ defines an ellipse and a hyperbola respectively. 2

(ii) Sketch the curve corresponding to the value of $k = 3$, showing foci, directrices and where the curve cuts the coordinate axes. 4

(iii) Describe how the shape of this curve changes as k varies from 3 to 7. 1

(c)



In ΔABC, BD bisects $\angle ABC$ as shown in the diagram.

(i) By considering the area of Δ ABC, show that 2

$$BD = \frac{2ac\cos x}{a + c}$$

(ii) Show that : $\cos x = \frac{1}{2}\sqrt{\frac{(a+c)^2 - b^2}{ac}}$ 2

(iii) Hence show that $BD = \frac{\sqrt{ac}}{a+c}\sqrt{(a+c)^2 - b^2}$ 1

QUESTION 8 (15 marks)

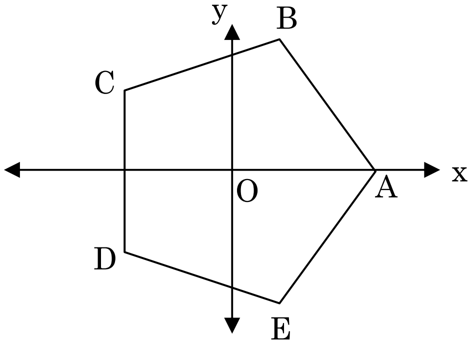
Marks

(a) If $I_n = \int_1^e x^3 (\log_e x)^n \, dx$ *for* $n = 0, 1, 2, \ldots$ show that

$$I_n = \frac{e^4}{4} - \frac{n}{4} I_{n-1}$$ and hence find the value of

$$\int_1^e x^3 (\log_e x)^2 \, dx$$ 4

(b)



In the diagram, the complex numbers z_0, z_1, z_2, z_3 *and* z_4 are represented by the vertices of a regular pentagon with center O and vertices A,B,C,D and E respectively. Given that $z_0 = 2$

(i) Express z_2 in modulus argument form 1

(ii) Find the value of $(z_2)^5$ 2

(iii) Show that the perimeter of the pentagon is $20 \sin \frac{\pi}{5}$ 2

(c) (i) Use De Moivre's Theorem to find expressions for $\cos 5\theta$ *and* $\sin 5\theta$ and hence show that: 3

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

(ii) By considering the equation: 3

$$x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0$$

Prove that $\tan\frac{\pi}{20} + \tan\frac{9\pi}{20} + \tan\frac{17\pi}{20} + \tan\frac{33\pi}{20} = 4$

EXT 2 TRIAL 2007

D a)

$\sin(A-B) + \sin(A+B) = \sin A\cos B - \sin B\cos A + \sin A\cos B + \sin B\cos A$

$= 2\sin A\cos B.$

let $A = 5x$ $\quad B = 3x$

$\therefore \int \sin 5x \cos 3x = \frac{1}{2}\int \sin 2x\,dx + \frac{1}{2}\int \sin 8x\,dx$

$= -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + c$

b) $(Ax+B)(x+1) + c(x^2+1) \equiv x^2 + 5x + 2$

$x = -1$ $\quad 2c = -2$ $\quad \therefore c = -1$

coeff of x^2 $\quad A + c = 1$ $\quad \therefore A = 2$

Constant term $\quad B + c = 2$ $\quad \therefore B = 3$

$\therefore I = \int \frac{2x+3}{x^2+1}\,dx - \int \frac{dx}{x+1}$

$= \ln(x^2+1) + 3\tan^{-1}x - \ln(x+1) + C$

c) $I = \int \sin^{-1}x\,dx$

let $u = \sin^{-1}x$ $\quad v' = 1$

$u' = \frac{1}{\sqrt{1-x^2}}$ $\quad v = x$

$\therefore I = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x + \sqrt{1-x^2} + C$

d) $\int \sqrt{\frac{1+x}{1-x}}\,dx$

$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx$

$= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$

$= \sin^{-1}x - \sqrt{1-x^2} + C$

e) $\int_0^1 x^5 e^{x^3}\,dx$

let $u = \frac{x^3}{3}$ $\quad v' = 3x^2 e^{x^3}$

$u' = x^2$ $\quad v = e^{x^3}$

$\therefore I = \left[\frac{x^3}{3}e^{x^3}\right]_0^1 - \int_0^1 x^2 e^{x^3}\,dx$

$= \frac{e}{3} - 0 - \left[\frac{e^{x^3}}{3}\right]_0^1$

$= \frac{e}{3} - \frac{e}{3} + \frac{1}{3}$

$= \frac{1}{3}$

② a) i) $|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2}$

$= \sqrt{12+4} = 4$

ii) $\sin\theta = \frac{\sqrt{3}}{2}$ $\cos\theta = -\frac{1}{2}$

$\therefore \arg z = \frac{2\pi}{3}$

iii) $Re\,(1+2i)(-2\sqrt{3}\,i-2)$

$= -2+4\sqrt{3}$

b) i) $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

if ω is a complex root $\omega \neq 1$

$\therefore \omega^2+\omega+1 = 0.$

ii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

now $\omega^4 = \omega^3.\omega = \omega$

$\omega^8 = (\omega^3)^2.\omega^2 . \omega^2$

$\therefore = (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$

$= (1-\omega-\omega^2+\omega^3)^2$

now $\omega+\omega^2 = -1$ $\omega^3 = 1$

$\therefore = (1+1+1)^2 = 9.$

c) $|z-2i| = \text{Im}\, z$

let $z = x+iy$

$\therefore \sqrt{x^2+(y-2)^2} = y$

$\therefore x^2+y^2-4y+4 = y^2$

$\therefore x^2-4y+4 = 0$

d) $z = 2-i$

$az + \frac{b}{z} = 1$

$\therefore a(2-i) + \frac{b}{2-i} = 1$

× bo by $(2-i)$

$a(2-i)^2+b = 2-i$

$a(3-4i) + b = 2-i$

$\therefore (3a+b-2)+i(-4a+1) = 0$

$\therefore a = \frac{1}{4}$ *

$\frac{3}{4}+b-2 = 0$

$\therefore b = \frac{5}{4}$ *

or $a(2-i) + \frac{b}{2-i}\cdot\frac{2+i}{2+i} = 1$

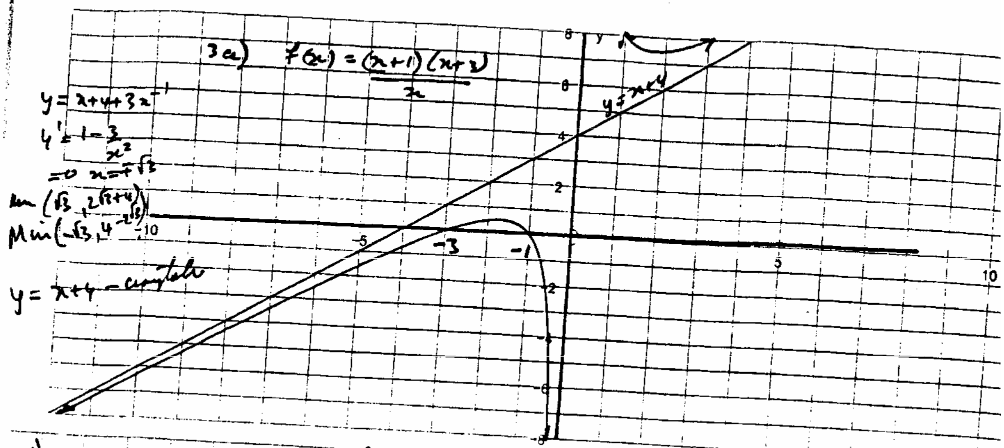
$2a - ai + \frac{2b+ib}{5} = 1$

$(10a+2b-5)+i(b-5a) = 0$

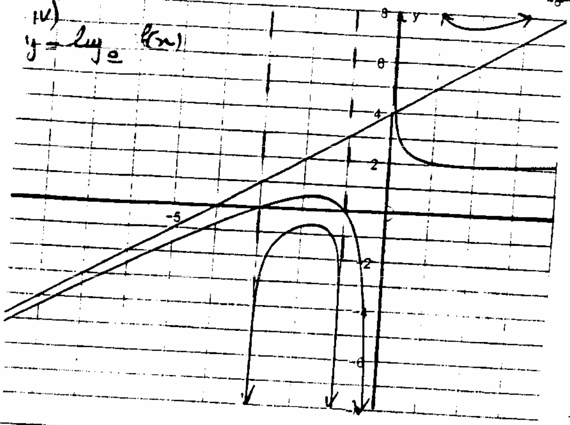
$10a+2b = 5$

$b-5a = 0$

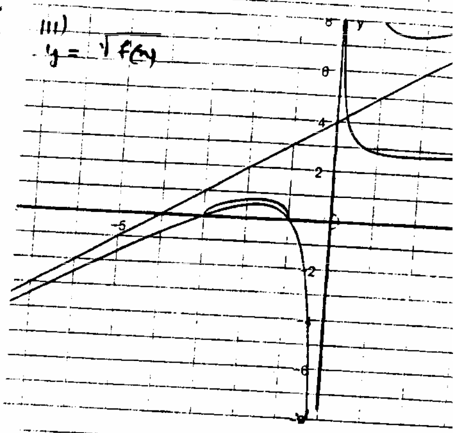
$\therefore b = \frac{5}{4}$ $a = \frac{1}{4}$



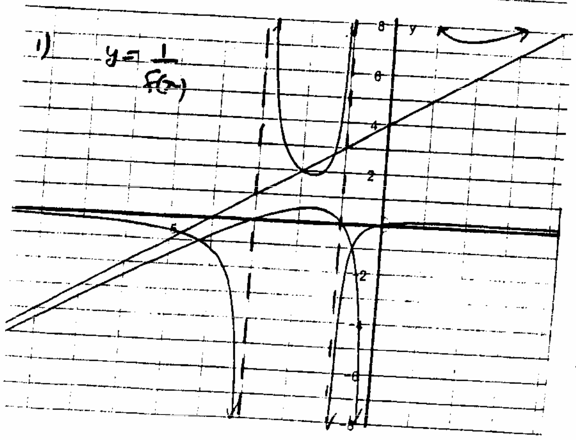
3a) $f(x) = \frac{(x+1)(x+3)}{x}$
$y = x+4+3x^{-1}$
$y' = 1 - \frac{3}{x^2}$
$= 0 \quad x = \pm\sqrt{3}$
max $(\sqrt{3}, 2(\sqrt{3}+4))$
Min $(-\sqrt{3}, 4-2\sqrt{3})$
$y = x+4$ – asymptote
$y = x+4$
-3
-1



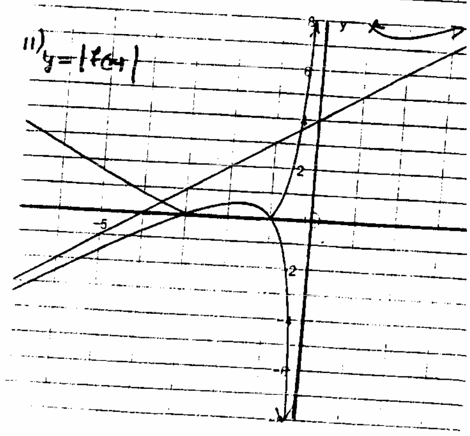
iv)
$y = \log_e f(x)$



iii)
$y = \sqrt{f(x)}$



i) $y = \frac{1}{f(x)}$



ii) $y = |f(x)|$

3b

let $x = a + bh$

when $h = 0$ $x = 3$, $h = 2, x = 5$

$\therefore \quad a = 3$

$x = 3 + bh$

$5 = 3 + 2b$

$\therefore b = 1$

$\therefore x = 3 + h.$

Similarly $y = 2 + \frac{h}{2}$

$\therefore \quad \delta A = (3+h)(2+\frac{h}{2})$

$\delta V = (3+h)(2+\frac{h}{2})\, \delta h$

$\therefore V = \int_0^2 (3+h)(2+\frac{h}{2})\, dh$

$= \frac{1}{2} \int_0^2 (12 + 7h + h^2)\, dh$

$= \frac{1}{2}\left[12h + \frac{7h^2}{2} + \frac{h^3}{3}\right]_0^2$

$= \frac{1}{2}\left\{24 + 14 + \frac{8}{3}\right\} - 0$

$= 20\frac{1}{3}$ cu units.

4a) i) $y = c^2 x^{-1}$

$y' = -\frac{c^2}{x^2} = -\frac{1}{p^2}$

∴ grad of normal is p^2

∴ N ⇒ $y - \frac{c}{p} = p^2(x - cp)$

∴ $p^3x - py = c(p^4 - 1)$

ii) Subst $x = cq$ $y = \frac{c}{q}$ ⇒ N

∴ $cp^3q - \frac{pc}{q} = cp^4 - c$

∴ $p^3q^2 - p = p^4q - q$

$p^3q^2 - p^4q = p - q$

$p^3q(q - p) = (p - q)$

∴ $p^3q = -1$

iii) equation of tangent is

$x + p^2y = 2cp$

subst $x = 0$ ∴ $y = \frac{2c}{p}$

∴ $R\left(0, \frac{2c}{p}\right)$

$P\left(cp, \frac{c}{p}\right)$

$Q\left(-\frac{c}{p^3}, -cp^3\right)$

Area of △ PQR = $\frac{1}{2}$ PR × QR

$= \frac{1}{2}\sqrt{(cp)^2 + \left(\frac{c}{p}\right)^2} \cdot \sqrt{\left(cp + \frac{c}{p^3}\right)^2 + \left(\frac{c}{p} + cp^3\right)^2}$

$= \frac{c^2}{2}\sqrt{p^2 + \frac{1}{p^2}} \cdot \sqrt{\left(p + \frac{1}{p^3}\right)^2 + \left(\frac{1}{p} + p^3\right)^2}$

$= \frac{c^2}{2}\sqrt{\frac{(p^4+1)}{p^2}} \cdot \sqrt{\left(\frac{p^4+1}{p^3}\right)^2 + \left(\frac{p^4+1}{p}\right)^2}$

$= \frac{c^2}{2}\sqrt{\frac{(p^4+1)}{p^2}} \cdot \sqrt{(p^4+1)^2\left(\frac{1}{p^6} + \frac{1}{p^2}\right)}$

$= \frac{c^2}{2}\frac{\sqrt{p^4+1}}{p} (p^4+1)\sqrt{\frac{p^4+1}{p^6}}$

$= \frac{c^2}{2}\frac{(p^4+1)^2}{p^4}$

$= \frac{c^2}{2}\left(\frac{p^4+1}{p^2}\right)^2$

$A = \frac{c^2}{2}\left(p^2 + \frac{1}{p^2}\right)^2$

$\frac{dA}{dp^2} = \frac{c^2}{2}\, 2\left(p^2 + \frac{1}{p^2}\right)\left(1 - \frac{1}{p^4}\right)$

$= 0$ when $p = \mp 1$

∴ $A = \frac{c^2}{2} \cdot 2^2 = 2c^2$.

b) $\int_0^{\frac{\pi}{2}} (\sin t)^{2k} \cos t \, dt$

$= \left[\frac{(\sin t)^{2k+1}}{2k+1}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2k+1}$

$\int_0^{\frac{\pi}{2}} (\cos t)^{2n+1} dt = \int_0^{\frac{\pi}{2}} \cos t (1 - \sin^2 t)^n dt$

$= \int {}^nC_0 \cos t - {}^nC_1 \cos t \sin^2 t + \ldots + (-1)^n \ldots$

$= \frac{1}{2\times0+1} - {}^nC_1 \frac{1}{2\times1+1} + {}^nC_2 \frac{1}{2\times2+1} \ldots$

$= \sum_{r=0}^{n} (-1)^r \frac{1}{2r+1} \cdot {}^nC_r$.

$\int_0^{\frac{\pi}{2}} \cos^7 t \, dt \longrightarrow n = 3$ (not ?

∴ $I = 1 - \frac{{}^3C_1}{3} + \frac{{}^3C_2}{5} - \frac{{}^3C_3}{7}$

Q5

(a) $x^2 - x + k = 0$

$\alpha + \beta = 1$, $\alpha\beta = k$

$\alpha, \beta > 0 \therefore k > 0$ ✓

α, β are real, distinct $\therefore \Delta > 0$

$1 - 4.1.k > 0$

$\frac{1}{4} > k$ ✓

(i) $\therefore 0 < k < \frac{1}{4}$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 1 - 2k$ ✓

Since $0 < k < \frac{1}{4}$

$0 > -2k > -\frac{1}{2}$ ✓

$1 > 1 - 2k > 1 - \frac{1}{2} = \frac{1}{2}$ ✓

$\therefore \alpha^2 + \beta^2 = 1 - 2k > \frac{1}{2}$

(iii) $k < \frac{1}{4}$, $\frac{1}{k} > 4$

$\frac{1}{k^2} > 4^2$, ie $\frac{1}{\alpha^2\beta^2} > 16$ ✓

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} > \frac{1}{2} \times 16$

> 8 ✓

(b) (i) Let $y = \frac{1}{x}$, $x = \frac{1}{y}$

$\therefore$ Eq is $\left(\frac{1}{y}\right)^3 - 7\left(\frac{1}{y}\right)^2 + 18\left(\frac{1}{y}\right) - 7 = 0$

$1 - 7y + 18y^2 - 7y^3 = 0$

or $7x^3 - 18x^2 + 7x - 1 = 0$ ✓✓

(ii) Let $y = 1 + x^2$

$x^2 = y - 1$ ✓

$\sqrt{y-1}(y-1) - 7(y-1) + 18\sqrt{y-1} - 7 = 0$

$\sqrt{y-1}(y+17) = 7y$

$(y-1)(y^2 + 34y + 289) = 49y^2$

$y^3 + 34y^2 + 289y - y^2 - 34y - 289 = 49y^2$

or $x^3 - 16x^2 + 255x - 289 = 0$

(c) $f(x) = x^3 - 3px^2 + 4q$

$f'(x) = 3x^2 - 6px$

S.P when $f'(x) = 0$ $3x(x - 2p) = 0$

$x = 0, 2p$

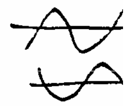
$f(0) = 4q > 0$ since $q > 0$

$f(2p) = (2p)^3 - 3p(2p)^2 + 4q$

$= 4q - 4p^3$

For 3 real distinct roots

product of y values < 0



$\therefore 4q - 4p^3 < 0$ since $4q > 0$

Q6

a) Downward $\downarrow +$

$\uparrow \frac{1}{10}mv^2$

$\downarrow mg$

$mg - \frac{1}{10}mv^2 = ma$

$\therefore a = g - \frac{1}{10}v^2$

terminal velocity when $a = 0$

$\therefore V^2 = 10g$

$V = \sqrt{10g}$

(ii) Upwards $\uparrow +$

$\downarrow \frac{1}{10}mv^2$ $\downarrow mg$

$-(mg + \frac{1}{10}mv^2) = ma$

$\therefore a = -\left(g + \frac{V^2}{10}\right)$

$v\frac{dv}{dx} = -\left(g + \frac{V^2}{10}\right)$

$-\int_{V\tan\alpha}^{0} \frac{10V}{10g + V^2}\, dv = \int_0^H dx$ ✓

$x = -5\ln(10g + v^2) + C$ $\quad C =$ $V\tan\alpha$

$\therefore H = \left[5\ln(10g + V^2)\right]_0^{V\tan\alpha}$

$= 5\ln(10g + V^2\tan^2\alpha) - 5\ln$

$= 5\ln \frac{10g + 10g\tan^2\alpha}{10g}$

$H =$

$H = 5\ln(1+\tan^2\alpha) = 5\ln\sec^2\alpha$ ✓

For downward, let speed at pt of projection be U

$v\frac{dv}{dx} = g - \frac{1}{10}v^2$

$\int_0^U \frac{10v}{10g-v^2}\,dv = \int_0^H dx$ ✓

$\left[-5\ln(10g-v^2)\right]_0^U = H$

$5\ln 10g - 5\ln(10g-U^2) = H$ ✓

$5\ln\sec^2\alpha = 5\ln\frac{10g}{10g-U^2}$

$10g - U^2 = \frac{10g}{\sec^2\alpha} = 10g\cos^2\alpha$

$U^2 = 10g(1-\cos^2\alpha)$

$= 10g\sin^2\alpha$ ✓

$= V^2\sin^2\alpha$

$U = V\sin\alpha$ ✓

(b) $4x^3 - 36x^2 + 107x + k = 0$

Let roots be $a-d, a, a+d$

Sum $= 3a = \frac{36}{4}$ $\therefore a = 3$ ✓

3 is a root

$\therefore 4(3)^3 - 36(3)^2 + 107(3) + k = 0$

$k = -105$ ✓

$$\begin{array}{r l}
 & 4x^2 - 24x + 35 \\
x-3 \,\big) & 4x^3 - 36x^2 + 107x - 105 \\
 & 4x^3 - 12x^2 \\
\hline
 & -24x^2 + 107x \\
 & -24x^2 + 72x \\
\hline
 & 35x - 105 \\
 & 35x - 105 \\
\hline
\end{array}$$

$\therefore (x-3)(4x^2-24x+35) = (x-3)(2x-7)(2x-5)$.

$x = \frac{5}{2}, 3, \frac{7}{2}$ ✓✓

b(a)

or

$x = -5\ln(10g-v^2) + c$

$c = 5\ln(10g)$ ✓

$x = 5\ln\frac{10g}{10g-v^2}$ ✓

$5\ln\sec^2\alpha = 5\ln\frac{10g}{10g-v^2}$

$\sec^2\alpha = \frac{10g}{10g-v^2}$

$10g - v^2 = 10g\cos^2\alpha$ ✓

$v^2 = 10g(1-\cos^2\alpha)$

$= V^2\sin^2\alpha$ ✓

$3a = \frac{36}{4}$ $a = 3$ ✓

$a^2 - \cancel{ad} + a^2 + \cancel{ad} + a^2 - d^2 = \frac{107}{4}$

$3a^2 - d^2 = \frac{107}{4}$

$d^2 = 27 - \frac{107}{4} = \frac{1}{4}$

$d = \pm\frac{1}{2}$ ✓

$\therefore$ Roots are $3-\frac{1}{2}, 3, 3+\frac{1}{2}$

$2\frac{1}{2}, 3, 3\frac{1}{2}$ ✓

$-\frac{k}{4} = \frac{5}{2}\times 3\times\frac{7}{2}$ $k = -105$ ✓

a) C mult by $i(4+3i)$

$\therefore$ C represents $-3+4i$

B mult by $(-3+4i)(4+3i)$

$= 1+7i$

b) i) for ellipse

$19-k>0$ and $7-k>0$

$k<19$ $\quad k<7$

$\therefore$ $k<7$ represents an ellipse

for hyp $19-k>0$ and $7-k<0$

$k<19$ $\quad k>7$

$\therefore$ ~~[illegible]~~ $7<k<19$

or $19-k<0$ $\quad 7-k>0$

$k>19$ $\quad k<7$

no soln

$\therefore$ $7<k<19$ represents a hyperbola

ii) $k=3 \Rightarrow$

$$\frac{x^2}{16}+\frac{y^2}{4}=1$$

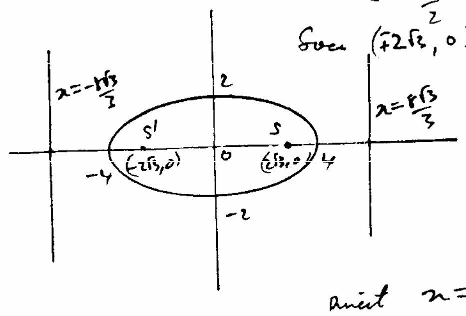
$a=4$ $\quad b=2$

$b^2=a^2(1-e^2)$

$4=16(1-e^2)$

$e=\frac{\sqrt{3}}{2}$

foci $(\mp 2\sqrt{3},0)$



direct $x=\frac{8}{\sqrt{3}}$

$x=\pm\frac{8}{3}\sqrt{3}$

c) i) AREA $= \frac{1}{2}.a.c \sin 2x$

also AREA $= \frac{1}{2}c.BD\sin x + \frac{1}{2}a BD \sin x$

$= \frac{BD \sin x}{2}(a+c)$

$\therefore \frac{BD\sin x(a+c)}{2} = \frac{1}{2}ac.2\sin x\cos x$

$\therefore BD = \frac{2ac\cos x}{a+c}$

ii) $\cos 2x = \frac{a^2+c^2-b^2}{2ac}$

$\therefore 2\cos^2 x - 1 = \frac{a^2+c^2-b^2}{2ac}$

$\cos^2 x = \frac{1}{2}\left(\frac{a^2+c^2-b^2}{2ac}+1\right)$

$= \frac{a^2+c^2-b^2+2ac}{4ac}$

$= \frac{(a^2+2ac+c^2)-b^2}{4ac}$

$= \frac{(a+c)^2-b^2}{4ac}$

$\therefore \cos x = \frac{1}{2}\sqrt{\frac{(a+c)^2-b^2}{ac}}$

iii) $BD = \frac{2ac\cos x}{a+c}$

$= \frac{2ac}{a+c}\cdot\frac{1}{2}\sqrt{\frac{(a+c)^2-b^2}{ac}}$

$= \frac{\sqrt{ac}\sqrt{(a+c)^2-b^2}}{a+c}$

8a) $I_n = \int_1^e x^3 (\ln x)^n \, dx$

let $u = (\ln x)^n \qquad v' = x^3$

$u' = \frac{n(\ln x)^{n-1}}{x} \qquad v = \frac{x^4}{4}$

$\therefore I_n = \left[\frac{x^4}{4}(\ln x)^n\right]_1^e - \frac{n}{4}\int_1^e x^3 (\ln x)^{n-1} \, dx$

$= \frac{e^4}{4} - 0 - \frac{n}{4} I_{n-1}$

$\therefore I_n = \frac{e^4}{4} - \frac{n}{4} I_{n-1}$

$\int_1^e x^3 (\ln x)^2 \, dx = I_2$

$I_2 = \frac{e^4}{4} - \frac{2}{4} I_1$

$I_1 = \frac{e^4}{4} - \frac{1}{4} I_0$

$I_0 = \int_1^e x^3 \, dx$

$= \left[\frac{x^4}{4}\right]_1^e$

$= \left(\frac{e^4-1}{4}\right)$

$I_1 = \frac{e^4}{4} - \frac{(e^4-1)}{16}$

$I_2 = \frac{e^4}{4} - \frac{1}{2}\left[\frac{e^4}{4} - \frac{(e^4-1)}{16}\right]$

$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{e^4}{32} - \frac{1}{32}$

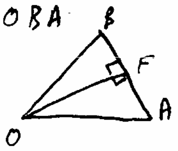
$= \frac{5e^4-1}{32}$

8b) i) $z_2 = 2 \operatorname{cis} \frac{4\pi}{5}$

ii) $(z_2)^5 = 2^5 \operatorname{cis} \frac{20\pi}{5}$

$= 32.$

iii) in $\triangle OBA$



Construct $OF \perp BA$

$\angle FOA = \frac{1}{2} \cdot \frac{2\pi}{5} = \frac{\pi}{5}$

$\sin \frac{\pi}{5} = \frac{FA}{OA} = \frac{FA}{2}$

$\therefore \quad FA = 2 \sin \frac{\pi}{5}$

$BA = 2FA = 4 \sin \frac{\pi}{5}$

$\therefore$ perimeter of pentagon $= 5 \times 4 \sin \frac{\pi}{5}$

$= 20 \sin \frac{\pi}{5}$.

c By De M. Th

$(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$

$= (c + is)^5$

$= c^5 + 5ic^4 s + 10 i^2 c^3 s^2 + 10 i^3 c^2 s^3 + 5 i^4 c s^4 + i^5 s^5$

$= (c^5 - 10c^3 s^2 + 5cs^4) + i(5c^4 s - 10c^2 s^3 + s^5)$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

$$\therefore \tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

$\div$ t+b by $\cos^5\theta$

$$\therefore \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

$x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0$

$x^5 - 10x^3 + 5x = 1 - 10x^2 + 5x^4$

$$\therefore \quad \frac{x^5 - 10x^3 + 5x}{1 - 10x^2 + 5x^4} = 1$$

let $x = \tan\theta \quad \therefore \tan 5\theta = 1$

$5\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}$

$\theta = \frac{\pi}{20}, \frac{5\pi}{20}, \frac{9\pi}{20}, \frac{13\pi}{20}, \frac{17\pi}{20}$

$\therefore x = \tan\frac{\pi}{20}, \tan\frac{\pi}{4}, \tan\frac{9\pi}{20}, \tan\frac{13\pi}{20}, \tan\frac{17\pi}{20}$

now $\tan\frac{\pi}{4} = 1 \qquad \tan\frac{13\pi}{20} = \tan\frac{33\pi}{20}$

$\therefore x = \tan\frac{\pi}{20}, \tan\frac{9\pi}{20}, \tan\frac{17\pi}{20}, \tan\frac{33\pi}{20}, 1$

from polynomial $\Sigma\alpha = 5$

$\therefore \tan\frac{\pi}{20} + \tan\frac{9\pi}{20} + \tan\frac{17\pi}{20} + \tan\frac{33\pi}{20} + 1 = 5$

$\therefore (\qquad\qquad) = 4$